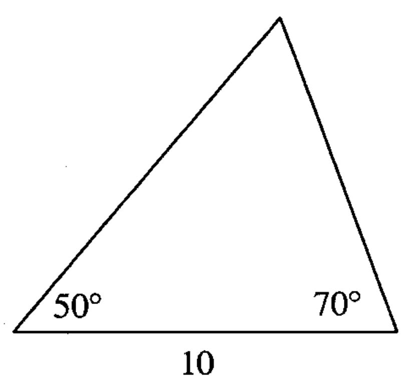
depending on whether the side is between or outside of the two angles). Another way to say this is to assert that the measure of two angles and one side determines the triangle. Geometry shows us one way to get the third angle (using the fact that the three angles of a triangle sum to 180°). But geometric methods do not let us compute the lengths of the other two sides. The law of sines allows us to do this.

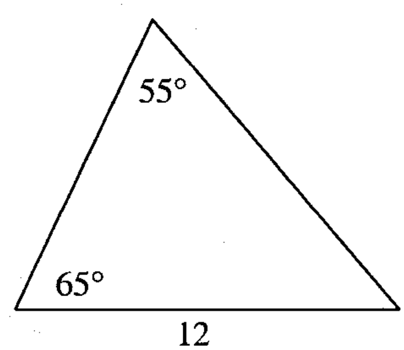
Exercises

1. Verify that the cyclic substitutions give the equalities shown above.

2. Check that the Law of Sines holds in a 30-60-90 triangle.

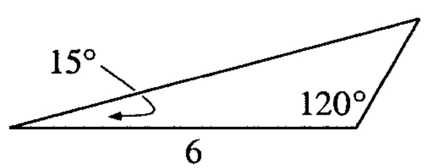
3. Use the Law of Sines in the triangles below to determine the lengths of the missing sides. (Use your calculator for the computations.)

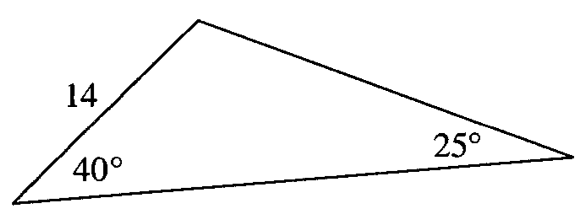




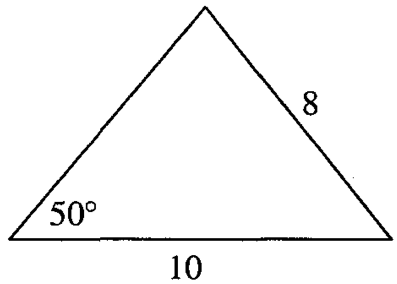
4. We have defined the sine of an obtuse angle as equal to the sine of its supplement. With this definition, show that the law of sines is true for an obtuse triangle.

5. Use the Law of Sines in the triangles below to determine the lengths of the missing sides.





6. Use the Law of Sines to find the two missing angles in the triangle below:



7. Recall from geometry that SSA does not guarantee congruence. That is, if two triangles match in two sides and an angle not included between these two sides, then the triangles may not be congruent. Look back at Problem 6. Is the triangle determined uniquely? How many possible values are there for the degree measurements of the remaining angles?

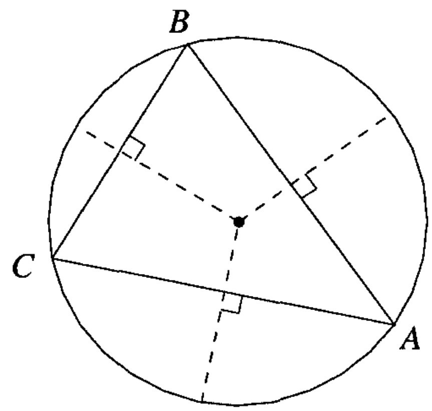
8. Suppose triangle ABC is inscribed in a circle of radius R. Prove the *extended Law of Sines*:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R\,.$$

6 The circumradius

We can learn more about the Law of Sines another way if we give a geometric interpretation of the ratio $a/\sin\alpha$ in any triangle ABC.

We construct the circle circumscribing the triangle:[2]



Suppose the radius of this circle (the *circumradius* of the triangle) is R. We know from the result on page 57 that

$$BC = a = 2R\sin\alpha\,.$$

[2]Recall that the perpendicular bisectors of the three sides of a triangle coincide at a point equidistant from all three vertices. This point is the center of the triangle's circumscribed circle.

So the ratio $a/\sin\alpha$ is simply equal to $2R$.

Exercises

1. Find the circumradius of a triangle in which a 30° angle lies opposite a side of length 10 units. Note that this information does not determine the triangle.

2. Find the circumradius of a 30-60-90 triangle with hypotenuse 8. Do you really need the result of this section to find this circumradius?

7 Area of a triangle

Our altitude formulas have given us one interesting result: the Law of Sines. We now show how they lead to a new formula for the area of a triangle. But in fact, the formula we present is not really new. It is just the usual formula from geometry, written in trigonometric form.

If S denotes the area of a triangle, we know that

$$S = \frac{1}{2}ah_a .$$

But $h_a = b\sin\gamma$, so we can write

$$S = \frac{1}{2}ab\sin\gamma .$$

This is our "new" formula. As with our other formulas, we can use "cyclic substitutions" (see page 69) to get two more formulas:

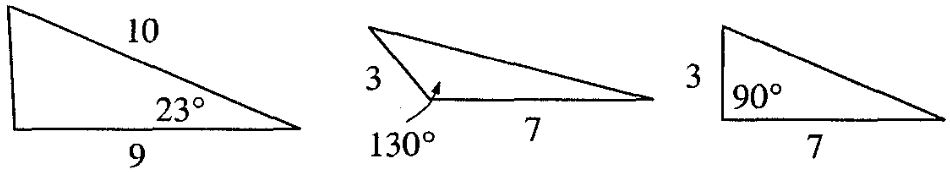
$$S = \frac{1}{2}bc\sin\alpha$$

$$S = \frac{1}{2}ca\sin\beta .$$

Exercises

1. Find the area of a triangle in which two sides of length 8 and 11 include an angle of 40° between them.

2. Find the areas of the triangles shown:



(a) (b) (c)

Can you use our new formula for part (c)? Is it necessary to use this formula?

3. The area of triangle ABC is 40. If side AB is 6 and angle A is 40 degrees, find the length of side AC.

4. In triangle PQR, side $PQ = 5$, and side $PR = 6$. If the area of the triangle is 9, find the degree-measure of angle P.

   Hint: There are two possible answers. Can you find them both?

5. Two sides of a triangle are a and b. What is the largest area the triangle can have? What is the shape of the triangle with largest area?

   Answer: The largest area is $ab/2$, achieved when the angle between the two sides is a right angle.

   Challenge: There is another right triangle with sides a and b. Find this triangle and its area.

6. The length of a leg of an isosceles triangle is x. Express in terms of x the largest possible area the triangle can have.

7. Show that the area of a parallelogram is $ab \sin C$, where a and b are two adjacent sides and C is one of the angles. Does it matter which angle we use?

8. We start with any quadrilateral whose diagonals are contained inside the figure. Show that the area of the quadrilateral is equal to half the product of the diagonals times the sine of the angle between the diagonals. Should we take the acute angle formed by the diagonals, or the obtuse angle?

9. Show that we can use the same formula to get the area of a quadrilateral whose diagonals (when extended) intersect outside the figure.